Algebra-IV, IInd Year BMath
Semestral Exam
May, 2022

Time: 3 hours

Marks: 50

(1) Prove or disprove.
(a) A finite extension of $\mathbb{Q}$ can contain only finitely many roots of unity.
(b) There exists a polynomial $f \in \mathbb{Z}[x]$ of degree greater than 1 that is irreducible modulo p for all primes p.
(c) If α is contained in a subfield of $\mathbb{R}$ that is Galois of degree $2r$ over $\mathbb{Q}$, then α is constructible.
(d) Galois group of the splitting field of the polynomial $x^4 - 2x^3 - 8x - 3$ over $\mathbb{Q}$ is A_3.
(e) $\mathbb{Q}\sqrt[3]{2})$ is not contained in any cyclotomic extension of $\mathbb{Q}$. (5× 5)

(2) Let p be a prime, and let m, n be positive integers. Give necessary and sufficient conditions on m and n for $\mathbb{F}_{p^n}$ to have a subfield isomorphic with $\mathbb{F}_{p^m}$. Prove your answer. (8)

(3) Let ζ be a primitive 7th root of unity, and consider the cyclotomic extension $K = \mathbb{Q}(\zeta)$ over $\mathbb{Q}$.
(a) Find the Galois group of K over $\mathbb{Q}$.
(b) Find all intermediate subextensions, i.e. all subfields $E \subseteq K$ such that E contains $\mathbb{Q}$.
(c) Find primitive elements for all subextensions E of part (b). Note that $\alpha \in E$ is a primitive element for E if $E = \mathbb{Q}(\alpha)$. (2+2+4)

(4) Let $f \in \mathbb{Q}[x]$ be a monic irreducible polynomial of degree n. Let K be a splitting field and let $G = \mathrm{Gal}(K/\mathbb{Q})$.
(a) Prove that G is a transitive subgroup of S_n with $o(G)$ divisible by n.
(b) Prove that if n is prime, then G contains an n-cycle.
(c) Suppose that degree of f is 5 and f has exactly 3 real roots. Prove that $G \cong S_5$. (3+2+4)